Int. J. Heat Mass Transfer. Vol. 9, pp. 165–170. Pergamon Press 1966. Printed in Great Britain

AN EXTENDED VARIATIONAL METHOD APPLIED TO POISEUILLE FLOW: TEMPERATURE DEPENDENT VISCOSITY

DONALD F. HAYS
General Motors Research Laboratories Warren, Michigan

(*Received* 21 *December* 1964 *and in revised form* 9 *July* 1965)

Abstract—The recent work of I. Prigogine and P. Glansdorff has shown that a variational method can be applied to problems which cannot be described by self-adjoint differential equations. As an example of the use of this extended variational principle, the problem of slow viscous incompressible flow through a tube is considered. The wall of the tube is maintained at a uniform temperature, and the thermal conductivity of the fluid is assumed to be constant. The steady state temperature and velocity distributions are determined over the tube radius for the particular case where the fluid viscosity is linearly dependent upon the temperature. Comparisons between the results obtained through numerical integration of the exact equations and those obtained using the variational approach are favorable over a range of the viscosity-temperature coefficient.

1. INTRODUCTION

THE EXTENDED variational formulation . was recently applied to both Couette and Poiseuille flow by the author [1]. The particular investigation was for the purpose of acquiring an understanding of the accuracy, efficiency, and general applicability of the methods of Glansdorff *et al.* [2], to fluid flow problems exhibiting mechanical irreversibilities due to viscous forces. In order to achieve an exact solution as a basis for comparison for the variational solution, a simplified but unrealistic form was assumed for the temperature dependence of the viscosity and thermal conductivity. The present paper illustrates more clearly the very general form which these terms may assume as a function of temperature.

For slow, viscous, incompressible flow, the variational form [2] to be set to zero is

$$\left.\begin{aligned}\delta \int_V \Bigg[\frac{1}{2}\left\{\frac{k_0 T_0}{T^2}\left(\frac{\partial T}{\partial x_i}\right)^2 + \frac{\mu_0 T_0}{T}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2\right\} \\ + \lambda \frac{\partial v_i}{\partial x_i}\Bigg] \mathrm{d}V + \int_\Omega \Bigg[\frac{-k_0 T_0}{T^2}\left(\frac{\partial T}{\partial x_i}\right)\delta T \\ + 2P_{ij}\delta v_j - 2\rho\omega\delta v_i\Bigg] n_i \,\mathrm{d}\Omega = 0\end{aligned}\right\} \quad (1.1)$$

where no conditions are imposed on the boundary surface Ω of the fluid volume V.

In equation (1.1), the temperature $T(x_i)$ is subject to variation, but the temperature distribution $T_0(x_i)$, which corresponds to the steady state, is not subject to variation. The viscosity μ_0 and thermal conductivity k_0 are both functions of the steady state temperature distribution $T_0(x_i)$. The remaining terms include the components of the fluid velocity v_i, the Lagrangian multiplier λ, the components of the stress tensor P_{ij}, a potential ω from which an external force is derived, the fluid density ρ, and n_i, the unit normal to the boundary surface Ω.

The formulation given by equation (1.1) has been applied to the slow flow of an incompressible fluid through a tube where the viscosity of the fluid is a linear function of the temperature. In this illustrative example, the thermal conductivity k is assumed to be constant. As in reference 1, comparatively simple forms are used to represent temperature and velocity distributions. Two numerical techniques are used to arrive at solutions from the variational formulation, and these results are compared with those obtained from the momentum and energy equations through numerical integration.

2. NUMERICAL SOLUTION

The flow system is that of Poiseuille flow through a tube of radius R. Cylindrical coordinates are used, z being parallel to the tube axis, r and θ being the radial and angular coordinates. The wall of the tube is at a constant temperature T_w, and the fluid driving force is a pressure gradient along the axis of the tube. Because of symmetry, $T = T(r)$ and $u = u(r)$ while $p = p(z)$. The momentum and energy equations are respectively

$$\frac{\mathrm{d}p}{\mathrm{d}Z} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\mu\frac{\mathrm{d}u}{\mathrm{d}r}\right), \tag{2.1}$$

$$0 = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rk\frac{\mathrm{d}T}{\mathrm{d}r}\right) + \mu\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2. \tag{2.2}$$

It is assumed that the viscosity variation with temperature is linear and that the thermal conductivity is constant. Thus

$$\mu = \mu^*[1 + \beta(T - T^*)], \quad k = \text{constant}. \tag{2.3}$$

In equation (2.3), T^* is a reference temperature which determines a viscosity μ^* and β is the viscosity–temperature coefficient. It is convenient to introduce the following dimensionless quantities into the momentum and energy equations:

$$\rho = \frac{r}{R}, \quad \theta = \frac{T}{T_w}, \quad \alpha = T_w\beta, \quad \nu = u\sqrt{\left(\frac{\mu^*}{kT_w}\right)}. \tag{2.4}$$

When these quantities are substituted into equation (2.1), and an integration is performed,

$$\rho\{1 + \alpha(\theta - \theta^*)\}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\gamma\rho^2}{2} + c_1, \tag{2.5}$$

where

$$\gamma = \frac{R^2}{\sqrt{(k\mu^* T_w)}}\left(\frac{\mathrm{d}p}{\mathrm{d}Z}\right) < 0$$

and c_1 is a constant of integration. Since $\mathrm{d}\nu/\mathrm{d}\rho = 0$ at $\rho = 0$, $C_1 = 0$ and

$$\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\gamma\rho}{2\{1 + \alpha(\theta - \theta^*)\}}. \tag{2.6}$$

In a similar manner, the energy equation (2.2) is reduced to the form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \frac{\gamma^2\rho^2}{4\{1 + \alpha(\theta - \theta^*)\}} = 0. \tag{2.7}$$

The dimensionless reference temperature θ^* may be arbitrarily chosen. In this instance it was chosen to be of the following form,

$$\theta^* = \frac{1 + \theta_m}{2} \tag{2.8}$$

where θ_m is the maximum dimensionless temperature which occurs at $\rho = 0$. By substituting (2.8) into equations (2.6) and (2.7), they are then given by the forms

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}\theta}{\mathrm{d}\rho} + \frac{\gamma^2\rho^2}{2\{2 + \alpha(2\theta - 1 - \theta_m)\}} = 0 \tag{2.9}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\gamma\rho}{2 + \alpha(2\theta - 1 - \theta_m)}. \tag{2.10}$$

The equations (2.9) and (2.10), with the boundary conditions

$$\theta(1) = 1, \quad \theta(0) = \theta_m, \quad \nu(1) = 0, \tag{2.11}$$

are solved for θ and ν through the use of a high speed digital computer. By assigning values for the fluid driving force γ, the viscosity coefficient α, and the maximum temperature θ_m, numerical integration of the two simultaneous differential equations (2.9) and (2.10) is accomplished using the Runga–Kutta–Gill technique. An iterative method is used, successive integrations being performed until the difference between the assigned value of θ_m and the value obtained by integrating over ρ is less than some arbitrarily small assigned quantity ϵ.

3. VARIATIONAL FORMULATION: SELF-CONSISTENT METHOD

A functional J which reflects the conditions of the variational form given by equation (1.1), and which is applicable to the one dimensional flow system being investigated, is as follows:

$$J = \int_0^{2\pi}\int_0^{R}\left[\frac{k_oT_o}{2T^2}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)^2 + \frac{\mu_oT_o}{T}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + 2\left(\frac{\mathrm{d}p}{\mathrm{d}Z}\right)u\right] r\mathrm{d}r\,\mathrm{d}\theta \tag{3.1}$$

The relationships given by (2.3) and (2.4) allow equation (3.1) to be written as

$$J = \int_0^1 \left[\frac{\theta_o}{2\theta^2}\left(\frac{d\theta}{d\rho}\right)^2 + \frac{\{1 + \alpha(\theta_o - \theta_o^*)\}\theta_o}{\theta}\left(\frac{d\nu}{d\rho}\right)^2 + 2\gamma\nu\right]\rho\, d\rho \quad (3.2)$$

where γ is defined in (2.5). The boundary conditions on the temperature and velocity are

$$\theta(1) = 1, \quad \theta(0) = \theta_m;$$
$$\nu(1) = 0, \quad \nu(0) = \nu_m, \quad (3.3)$$

where θ_m and ν_m are the maximum values of the temperature and velocity. It should be noted that the temperature functions θ_o in equation (3.2) which represent the stationary state are not to be varied but they must be the same function of ρ as the variable θ which is to be varied.

Elementary forms for functions of temperature and velocity which satisfy the conditions of (3.3) are

$$\theta = (1 - \rho^4)A + 1, \quad \nu = (1 - \rho^2)B,$$
$$\theta_o = (1 - \rho^4)A_o + 1, \quad (3.4)$$

where $\theta_m = A + 1$ and $\nu_m = B$.

Because of the illustrative nature of this example, only one arbitrary constant has been incorporated into the functions of (3.4), these being A and B. Additional constants would give a greater flexibility to the approximating functions, but at the cost of increasing the complexity of the algebra of the problem. The values of A and B which allow the best solution of the basic equations by the forms of θ and ν given by (3.4) are determined from the relationships

$$\frac{\partial J}{\partial A} = 0, \quad \frac{\partial J}{\partial B} = 0. \quad (3.5)$$

In the self-consistent method, following the operations of (3.5), θ_o is set equal to θ, i.e. $A_o = A$. Thus,

$$\left.\begin{aligned}\frac{\partial J}{\partial A} = 0 = \int_0^1 \Bigg[\theta_o\bigg\{\frac{1}{\theta^2}\left(\frac{d\theta}{d\rho}\right)\frac{d}{dA}\left(\frac{d\theta}{d\rho}\right) \\ - \frac{1}{\theta^3}\left(\frac{d\theta}{dA}\right)\left(\frac{d\theta}{d\rho}\right)^2\bigg\} - \frac{\{1 + \alpha(\theta_o - \theta_o^*)\}\theta_o}{\theta^2} \\ \left(\frac{d\theta}{dA}\right)\left(\frac{d\nu}{d\rho}\right)^2\Bigg]\rho\, d\rho.\end{aligned}\right\} (3.6)$$

Now setting $\theta_o = \theta$, and using the relationships given by (2.8) and (3.4), the integration of equation (3.6) yields the following expression:

$$2B^2(1 - \alpha) - \frac{1}{A}\{8A + \alpha B^2(A + 2) - 2B^2\} \ln\left(\frac{1}{A + 1}\right) - 8A = 0. \quad (3.7)$$

In a similar manner,

$$\frac{\partial J}{\partial B} = 0 = \int_0^1 \left[\frac{\{1 + \alpha(\theta_o - \theta_o^*)\}\theta_o}{\theta}\left(\frac{d\nu}{d\rho}\right) \frac{d}{dB}\left(\frac{d\nu}{d\rho}\right) + \gamma\left(\frac{d\nu}{dB}\right)\right]\rho\, d\rho. \quad (3.8)$$

After setting $\theta_o = \theta$ and integrating, equation (3.8) reduces to the simple form,

$$B = \frac{-\gamma}{4}. \quad (3.9)$$

This direct proportionality of the maximum flow velocity to the fluid driving force, $B = B(\gamma)$, and its independence of viscosity and temperature effects is a consequence of the assumed form for $u(B, \rho)$ and the particular choice of the reference temperature T^*.

The substitution of (3.9) into (3.7) yields the following equation which must be solved for A,

$$\gamma^2(1 - \alpha) - 64A - \frac{1}{A}\left[64A + \frac{\alpha\gamma^2(2 + A)}{2} - \gamma^2\right] \ln\left(\frac{1}{A + 1}\right) = 0. \quad (3.10)$$

The values of α and γ are assigned and the value of A which satisfies (3.10) is then found through the use of an interval halving technique with a digital computer. With A and B known, equations (3.4) are used to obtain the temperature and velocity distributions.

4. VARIATIONAL FORMULATION: ITERATION METHOD

It is possible to solve for the coefficients in the variational formulation by using an iterative technique rather than the self-consistent method. The Newton–Raphson method [3] was found to be adequate for this analysis where the functional

J is a function of the three coefficients A, A_o, and B. This method is applied to the current analysis in the following manner.

As a consequence of the Ritz–Galerkin method, the two equations of (3.5) are established with the subsidiary conditions that at the steady state condition $\theta(\rho) = \theta_o(\rho)$. Thus, the following three equations are available:

$$\left.\begin{aligned} \frac{\partial J}{\partial A} &= F(A_o, A, B) = 0 \\ \frac{\partial J}{\partial B} &= G(A_o, A, B) = 0 \\ H &= A_o - A = 0 \end{aligned}\right\} \quad (4.1)$$

These functions are expanded in a Taylor's series and by retaining only the first-order terms, the following set of equations is obtained:

$$\left.\begin{aligned} F_{A_o}\delta A_o &= F_A\delta A + F_B\delta B = -F(A_o, A, B) \\ G_{A_o}\delta A_o + G_A\delta A + G_B\delta B &= -G(A_o, A, B) \\ H_{A_o}\delta A_o + H_A\delta A &= -H(A_o, A) \end{aligned}\right\} \quad (4.2)$$

where subscripts on F, G, and H denote differentiation. These equations are now solved for δA_o, δA, and δB.

With an initial estimate of A_o, A, and B, the change in these variables δA_o, δA, and δB, is then determined. Succeeding corrected values are obtained from the relationships

$$\left.\begin{aligned} A_o^{(n+1)} &= A_o^{(n)} + \delta A_o \\ A^{(n+1)} &= A^{(n)} + \delta A \\ B^{(n+1)} &= B^{(n)} + \delta B \end{aligned}\right\} \quad (4.3)$$

where n indicates the iteration number. This iteration process is continued until the difference between successive iterations is sufficiently small, i.e. until $\delta(\ ) < \epsilon$.

5. RESULTS

Numerical solutions were obtained for the temperature and velocity distribution across the radius. The numerical solution of the exact equations (2.9) and (2.10) for the terms $\theta_m - 1$ and ν_m, which correspond to the variables A and B, yielded values given in Table 1 over a range of α and γ.

For the self-consistent method, B is given directly by equation (3.9) and values of A which satisfy equation (3.10) over a range of α and γ are given in Table 2.

The Newton–Raphson iterative technique given by equations (3.3) and (3.4) yielded values of A and B, which differed by at most one figure in the fifth decimal place with the values obtained by the self-consistent method.

Curves of temperature and velocity across the tube radius are shown in Fig. 1 and 2 for a constant value of γ and for two values of α. It is seen from these curves that the difference between the results of the variational method and the exact equations increase with an increase in α. The discrepancy in the results can be attributed to the choice of the trial functions selected to represent the solutions of the differential equations. A trial function with a greater number of coefficients would have a greater flexibility in approximating the true solutions, but at the expense of increased analytical complexity. This difference, however, is slight; the percentage error in ν_m for a γ of $-3{\cdot}0$ and an

Table 1. Values of $\theta_m - 1$ and ν_m (comparable to A and B of the variational method) obtained by the numerical integration of the momentum and energy equations

	$\theta_m - 1$				ν_m			
γ \ α	0	−0·5	−1·0	−2·0	0	−0·5	−1·0	−2·0
−1·0	0·015625	0·01566	0·01569	0·01575	0·2500	0·25033	0·25066	0·25133
−2·0	0·062500	0·06300	0·06351	0·06460	0·5000	0·50267	0·50546	0·51149
−3·0	0·140625	0·14320	0·14600	0·15244	0·7500	0·75928	0·76963	0·79444
−4·0	0·250000	0·25841	0·26826	0·29492	1·0000	1·02300	1·05116	1·13320
−5·0	0·390625	0·41210	0·44028	0·55305	1·2500	1·29760	1·36471	1·67670

Table 2. Values of A and B (comparable to $\theta_m - 1$ and ν_m of the exact equations) obtained from the variational formulation using the self-consistent method

	A				*B*			
γ \ α	0	$-0{\cdot}5$	$-1{\cdot}0$	$-2{\cdot}0$	0	$-0{\cdot}5$	$-1{\cdot}0$	$-2{\cdot}0$
$-1{\cdot}0$	0·015625	0·01560	0·01559	0·01555	0·250	0·250	0·250	0·250
$-2{\cdot}0$	0·062500	0·06218	0·06187	0·06125	0·500	0·500	0·500	0·500
$-3{\cdot}0$	0·140625	0·13907	0·13754	0·13458	0·750	0·750	0·750	0·750
$-4{\cdot}0$	0·250000	0·24526	0·24069	0·23000	1·000	1·000	1·000	1·000
$-5{\cdot}0$	0·390626	0·37958	0·36910	0·34960	1·250	1·250	1·250	1·250

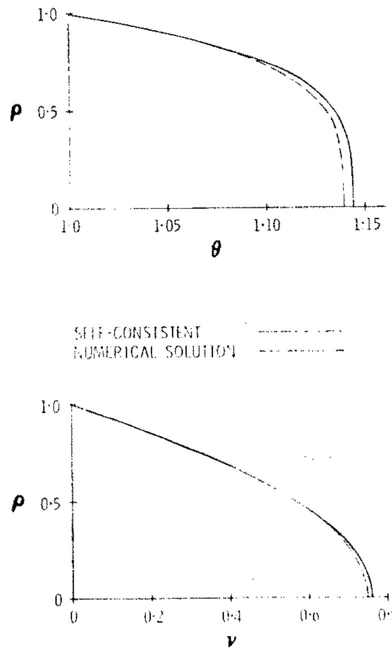

FIG. 1. Dimensionless temperature θ and velocity ν vs. tube radius ρ for both self-consistent method and numerical solution. $\alpha = -0{\cdot}5$, $\nu = -3{\cdot}0$.

α of $-2{\cdot}0$ being 5·6% and for θ_m the error is but 1·5%. Thus, over a reasonable range of the forcing term γ, the variational formulation yields quite acceptable results with a minimum number of coefficients. The results of the iteration technique have not been plotted since they match so closely the results of the self-consistent method. In applying the iterative method to the determination of the coefficients A and B, some care must be taken in choosing the initial values of these terms. If these values are too far removed from steady state values, then the iterative process may diverge. The range over which initial values of A and B will yield a converging process is a function of the derivatives J_A and J_B and is generally not easily determined. Consequently, some time may be required in searching for acceptable trial values. However, once the converging iterative process was initiated, convergence was achieved in from four to sixteen iterations with an ϵ of 10^{-6}.

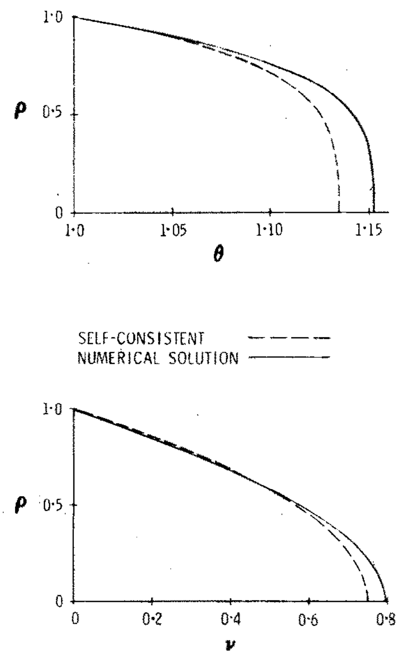

FIG. 2. Dimensionless temperature θ and velocity ν vs. tube radius ρ for both self-consistent method and numerical solution. $\alpha = -2{\cdot}0$, $\nu = -3{\cdot}0$.

Of greater importance than the actual values obtained in this problem is the realization that the variational formulation furnishes a method by which temperature dependent viscosity and thermal conductivity can be incorporated easily into fluid flow analyses. The complexity of the temperature dependence of these functions does not limit the use of this technique nor does the complexity of the assumed temperature and velocity distributions act as restraints upon its applicability. With the availability of high-speed digital computers and a general theory relating to the determination of a stationary states through a variational principle [4], approximate solutions of sufficient accuracy may be obtained for many of the non-isothermal fluid mechanics problems which may include such phenomena as diffusion and chemical reactions.

REFERENCES

1. D. F. Hays, A variational formulation applied to Couette and Poiseuille flow, Bulletin de l'Academie royale de Belgique (Classe des Sciences) Seance du 8 Juin, 1963, pp. 576–602.
2. P. Glansdorff, I. Prigogine and D. F. Hays, "Variational Properties of a Viscous Liquid at a Non-uniform Temperature," *Physics Fluids* **5**, 2, 144–149 (1962).
3. A. D. Booth, *Numerical Methods*, Academic Press, New York (1955).
4. P. Glansdorff and I. Prigogine, On a general evolution criterion in macroscopic physics, *Physica*, *'s Grav*. **30**, 351–374 (1964).

Résumé—Le travail récent de I. Prigogine et P. Glansdorff a montré qu'une méthode variationnelle peut être appliquée aux problèmes qui ne peuvent pas être décrits par des équations différentielles auto-adjointes. Comme exemple de l'utilisation de ce principe variationnel généralisé, le problème d'un écoulement incompressible visqueux lent à travers un tube est considéré. La paroi du tube est maintenue à une temérature uniforme, et l'on suppose que la conductivité thermique du fluide est constante. Les distributions de température et de vitesse en régime permanent sont déterminées le long du rayon du tube dans le cas particulier où la viscosité du fluide dépend linéairement de la température. Des comparaisons entre les résultats obtenus grâce à l'intégration numérique des équations exactes et ceux obtenus en utilisant la méthode variationnelle sont favorables dans une certaine gamme du coefficient de dépendance de la viscosité en fonction de la température.

Zusammenfassung—Die jüngste Arbeit von J. Prigogine und P. Glansdorff hat gezeigt, dass eine Variationsmethode auf Probleme angewandt werden kann, die von selbst angleichenden Differentialgleichungen nicht beschrieben werden können. Als Anwendungsbeispiel für dieses erweiterte Variationsprinzip wird das Problem der langsamen, zähen, inkompressiblen Strömung durch ein Rohr betrachtet. Die Rohrwand wird auf gleichmässiger Temperatur gehalten und die Wärmeleitfähigkeit der Flüssigkeit wird als konstant angenommen. Die Beharrungstemperatur und die Geschwindigkeitsverteilungen wurden über den Rohrradius für den besonderen Fall bestimmt, dass die Zähigkeit linear abhängig von der Temperatur ist. Ein Vergleich der Ergebnisse, die durch numerische Integration der exakten Gleichungen erhalten wurden, mit den nach der Variationsnäherung errechneten ist vielversprechend über einen Bereich des Zähigkeits-Temperatur-Koeffizienten.

Аннотация—В последней работе И. Пригожина и П. Гленсдорфа показано, что вариационный метод можно применить к задачам, которые нельзя описать с помощью самосопряженных дифференциальных уравнений. В качестве примера применения этого обобщенного вариационного принципа рассмотрена задача о медленном вязком несжимаемом течении жидкости в трубе. Температура стенки поддерживается постоянной, теплопроводность жидкости также принята постоянной. В стационарном состоянии определены распределения скорости и температуры по радиусу трубы для частного случая линейной зависимости вязкости жидкости от температуры. Сравнение результатов, полученных численным интегрированием точных уравнений, и данных, найденных с помощью вариационного метода, показало хорошее соответствие в диапазоне изменения вязкостно-температурного коэффициента.